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FAST TRACK COMMUNICATION

One-dimensional anyons with competing δ -function and derivative δ -function potentials

M T Batchelor^{1,2}, X-W Guan¹ and A Kundu³¹ Department of Theoretical Physics, Research School of Physical Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia² Mathematical Sciences Institute, Australian National University, Canberra ACT 0200, Australia³ Saha Institute of Nuclear Physics, Theory Group, CAMCS 1/AF Bidhan, Calcutta 700 064, IndiaE-mail: Murray.Batchelor@anu.edu.au, xwe105@rsphysse.anu.edu.au and anjan.kundu@saha.ac.in

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Online at stacks.iop.org/JPhysA/41/352002**Abstract**

We propose an exactly solvable model of one-dimensional anyons with competing δ -function and derivative δ -function interaction potentials. The Bethe ansatz equations are derived in terms of the N -particle sector for the quantum anyonic field model of the generalized derivative nonlinear Schrödinger equation. This more general anyon model exhibits richer physics than that of the recently studied one-dimensional model of δ -function interacting anyons. We show that the anyonic signature is inextricably related to the velocities of the colliding particles and the pairwise dynamical interaction between particles.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Anyons—which interpolate between bosons and fermions—may exist in two dimensions (2D), obeying fractional statistics. Fractional statistics have recently been observed in experiments on the quasiparticle excitations of a 2D electron gas in the fractional quantum Hall (FQH) regime [1]. The anyonic quasiparticles have profound implications for topological quantum states of matter [2]. For example, the non-Abelian topological order can be studied through manipulating FQH states in a 2D electron gas. As a consequence, the concept of anyons has become important in topological quantum computation [3, 4]. In one dimension (1D) anyons acquire a step-function-like phase when two identical particles exchange their positions.

Different aspects of anyons in 1D have been considered [5–7]. Among these an anyonic extension [8] of the 1D integrable Bose gas with δ -function interaction [9, 10] has attracted recent attention, including the basic construction of the anyon model [11–13]. Many new results have been obtained for this model, including the connection with Haldane [14] exclusion statistics [15, 16], correlation functions [12, 17–19], entanglement [20] and expanding anyonic fluids [21]⁴.

A key feature of 1D anyons is that they retain fractional statistics in quasi-momentum space and the anyonic signature of the exchange interaction—the topological anyonic interaction and dynamical interaction are inextricably related. 1D anyons show an intriguing sensitivity of the boundary condition of their wavefunction on the specific position of the particles [11, 12, 23]. Therefore the study of exactly solvable interacting anyons in 1D should give insight into understanding topological effects in many-body physics.

In this communication we propose a 1D model of anyons with δ -function and derivative δ -function interaction and solve the model by means of the coordinate Bethe ansatz. The model contains an additional independent interaction parameter so that in principle one can study anyonic signatures through tuning the interactions. For this more general model we show that the anyonic signature is inextricably related to the velocities of the colliding particles and the pairwise dynamical interaction between identical particles.

2. The model

We consider creation and annihilation operators $\Psi^\dagger(x)$ and $\Psi(x)$ at point x , which satisfy the anyonic commutation relations

$$\begin{aligned}\Psi(x)\Psi^\dagger(y) &= e^{-i\kappa w(x,y)}\Psi^\dagger(y)\Psi(x) + \delta(x-y) \\ \Psi^\dagger(x)\Psi^\dagger(y) &= e^{i\kappa w(x,y)}\Psi^\dagger(y)\Psi^\dagger(x) \\ \Psi(x)\Psi(y) &= e^{i\kappa w(x,y)}\Psi(y)\Psi(x).\end{aligned}\quad (1)$$

The multi-step function $w(x, y)$ appearing in the phase factors satisfies $w(x, y) = -w(y, x) = 1$ for $x > y$, with $w(x, x) = 0$. In terms of these operators the Hamiltonian describing N anyons of atomic mass m confined in a length L is

$$\begin{aligned}H &= \frac{\hbar^2}{2m} \int_0^L dx \Psi_x^\dagger(x)\Psi_x(x) + \frac{1}{2}g \int_0^L dx \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x) \\ &\quad + i\epsilon \int_0^L dx [\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi_x(x) - \Psi_x^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)],\end{aligned}\quad (2)$$

where we also impose periodic boundary conditions.

The tunable parameters in the model are (i) $g = \hbar^2 c/m$, the dynamical interaction strength which drives the particles in elastic scattering through zero range contact interaction, where c is the coupling constant, (ii) the nonlinear dispersion coupling constant ϵ , which describes a delta shift in collisions, and (iii) the statistical parameter κ which varies the statistics of the particles from pure Bose statistics ($\kappa = 0$) to pure Fermi statistics ($\kappa = \pi$). The model differs from the interacting 1D Bose gas [9, 10] and 1D interacting anyons [8] due to the presence of the nonlinear dispersion ϵ , i.e., the s-wave scattering is disturbed by a delta shift in collisions. Effectively, the 1D scattering length is increased or decreased in individual pairwise collisions by this dispersion. It follows that the model contains well-known models as special cases in certain limits. For $\kappa = 0$, Hamiltonian (2) reduces to that of the generalized nonlinear Schrödinger model [24, 26, 27], which has been studied by means of scattering Bethe ansatz

⁴ Further discussion of the ground-state properties is given in [22].

in the context of soliton physics. For $\epsilon = 0$, it describes the model of 1D interacting anyons. For $\kappa = \epsilon = 0$, it reduces to the model of 1D interacting bosons, while for $\kappa = c = 0$ it reduces to the quantum derivative nonlinear Schrödinger model [28]. As we shall see, the general model (2) exhibits more exotic dynamics than the two models corresponding to the special cases $\kappa = 0$ and $\epsilon = 0$.

For convenience, we hereafter set $\hbar = 2m = 1$. We also use a dimensionless coupling constant $\gamma = c/n$ to characterize different physical regimes of the anyon gas, where $n = N/L$ is the linear density.

3. Bethe ansatz

We first present the corresponding equation of motion $-i\partial_t\Psi(x, t) = [H, \Psi(x, t)]$ via the time-dependent quantum fields, i.e., the nonlinear Schrödinger equation is given by

$$i\Psi_t = -\Psi_{xx} + g\Psi^\dagger\Psi\Psi + 4i\epsilon\Psi^\dagger\Psi\Psi_x. \quad (3)$$

We note that this is an anyonic version of the generalized nonlinear Schrödinger equation. Restricting to its N -particle sector we can reduce the eigenvalue problem of the Hamiltonian (2) to a quantum many-body problem. To this end, we define a Fock vacuum state $\Psi(x)|0\rangle = 0$. Thus we can prove that the number operator N and momentum operator P are conserved, where

$$N = \int_0^L dx \Psi^\dagger(x)\Psi(x) \quad (4)$$

$$P = i \int_0^L dx \Psi_x^\dagger\Psi(x). \quad (5)$$

In order to properly assign the anyonic phase $w(x_i, x_j)$ [8, 11, 12, 15] in the domain $x_1 < x_2 < \dots < x_N$, we write the N -particle eigenstate as

$$|\Phi\rangle = \int_0^L dx^N e^{-i\frac{1}{2}\kappa N} \chi(x_1 \dots x_N) \Psi^\dagger(x_1) \dots \Psi^\dagger(x_N)|0\rangle, \quad (6)$$

where the wavefunction amplitude is of the form

$$\chi(x_1 \dots x_N) = e^{-i\frac{1}{2}\kappa \sum_{x_i < x_j} w(x_i, x_j)} \sum_P A(k_{P1} \dots k_{PN}) e^{i(k_{P1}x_1 + \dots + k_{PN}x_N)}. \quad (7)$$

Here the sum extends over all $N!$ permutations P . The choice of the sign of the anyonic phase factor in equation (7) can be arbitrary and may lead to different boundary conditions [12]. Here we prefer to fix a boundary condition where we count the anyonic phase factor in equation (7) through the order of the particles in the scattering states [11]. By acting on the eigenstate (6) with Hamiltonian (2) the eigenvalue problem for Hamiltonian (2), namely

$$H|\Phi\rangle = \int_0^L dx^N e^{-i\frac{1}{2}\kappa N} H_N \chi(x_1 \dots x_N) \Psi^\dagger(x_1) \dots \Psi^\dagger(x_N)|0\rangle \quad (8)$$

can be reduced to solving the quantum-mechanical problem

$$H_N \chi(x_1 \dots x_N) = E \chi(x_1 \dots x_N), \quad (9)$$

where

$$H_N = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g \sum_{1 \leq i < j \leq N} \delta(x_i - x_j) + i2\epsilon \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)(\partial_{x_i} + \partial_{x_j}). \quad (10)$$

This anyon model turns out to be Bethe ansatz solvable with independent values of c , ϵ and κ . Changing the coordinates to centre-of-mass coordinates $X = (x_j + x_k)/2$ and $Y = x_j - x_k$ leads to the eigenvalue equation (9) in the form

$$\left\{ \left(-\frac{\partial^2}{\partial x_1^2} \cdots - \frac{1}{2} \frac{\partial^2}{\partial X^2} - 2 \frac{\partial^2}{\partial Y^2} \cdots - \frac{\partial^2}{\partial x_N^2} \right) + 2c\delta(Y) + i4\epsilon\delta(Y)\partial_X - E \right\} \chi(\dots x_i \dots x_j \dots) = 0. \quad (11)$$

Integrating both sides of this equation with respect to Y from $-e$ to $+e$ and taking the limit $e \rightarrow 0$ gives the discontinuity condition

$$(\partial_{x_j} - \partial_{x_i})\chi(x_1, \dots, x_i, x_j, \dots, x_N)|_{x_j=x_i+e} - (\partial_{x_j} - \partial_{x_i})\chi(x_1, \dots, x_j, x_i, \dots, x_N)|_{x_j=x_i-e} = [2c + i2\epsilon(\partial_{x_i} + \partial_{x_j})]\chi(x_1, \dots, x_i, x_j, \dots, x_N)|_{x_i=x_j} \quad (12)$$

on the derivative of the wavefunction. This equation gives a relation between the coefficients $A(k_{P1} \dots k_{PN})$ of the form

$$A(\dots k_j, k_i \dots) = \frac{k_j - k_i + i(c' - \epsilon'(k_j + k_i))}{k_j - k_i - i(c' - \epsilon'(k_j + k_i))} A(\dots k_i, k_j \dots) \quad (13)$$

which is the two-body scattering relation. This relation gives rise to a functional scalar scattering matrix which factorizes three-body scattering processes into the product of three two-body scattering matrices [24]. In this sense the Yang–Baxter equation is trivially satisfied, guaranteeing no diffraction in the scattering process [10, 24]. Consequently, the N -particle scattering amplitude can be factorized into the products of two-particle amplitudes. Another way of establishing the integrability of the model is to explicitly construct an infinite number of higher order conserved quantities, as shown for the 1D Bose gas in [25].

The effective interaction strengths c' and ϵ' appearing in the solution are given by

$$c' = \frac{c}{\cos(\kappa/2)} \quad \text{and} \quad \epsilon' = \frac{\epsilon}{\cos(\kappa/2)}. \quad (14)$$

In this model these effective coupling constants implement the transmutation between statistical and dynamical interactions.

To complete the solution in terms of the Bethe ansatz we impose the periodic boundary condition $\chi(x_1 = 0, x_2, \dots, x_N) = \chi(x_2, \dots, x_N, x_1 = L)$ on the anyonic wavefunction in the fundamental region $0 \leq x_1 < x_2 < \dots < x_N \leq L$. As in the solution of the interacting anyon model ($\epsilon = 0$), this leads to the appearance of various phase factors [11, 12]. The wavefunctions for other regions can be determined through application of the anyonic symmetry. The periodic boundary condition can also be imposed on other wavefunction arguments x_i , in each case leading to the same anyonic phase factor in the Bethe ansatz equations (BAE) [11, 12]. For this model, the energy is given by

$$E = \sum_{i=1}^N k_i^2, \quad (15)$$

where the quasi-momenta k_j satisfy the BAE

$$e^{ik_j L} = -e^{i\kappa(N-1)} \prod_{\ell=1}^N \frac{k_j - k_\ell + i(c' - \epsilon'(k_j + k_\ell))}{k_j - k_\ell - i(c' - \epsilon'(k_j + k_\ell))} \quad (16)$$

for $j = 1, \dots, N$. The total momentum is

$$p = \sum_{i=1}^N k_i = N(N-1)\kappa/L. \quad (17)$$

In minimizing the energy we consider $\kappa(N-1) = \nu \pmod{2\pi}$ in the phase factor.

The Bethe ansatz solution provides in principle the full physics of the model. For the case $\kappa = 0$, the BAE (16) are consistent with the scattering states constructed for the generalized nonlinear Schrödinger model [24, 26]. It is also clear to see from the BAE (16) that the phase shift

$$\theta_{j,\ell} = \ln \left[\frac{k_j - k_\ell + i(c' - \epsilon'(k_j + k_\ell))}{k_j - k_\ell - i(c' - \epsilon'(k_j + k_\ell))} \right] \quad (18)$$

depends essentially on the anyon parameter κ , the dynamical interaction strength c and the nonlinear dispersion of velocity ϵ through the effective interactions (14). Among the competing interactions, the nonlinearity parameter ϵ is essential in determining the bound states. The δ -function interaction strength c plays the role of a driving force in collisions. However, the anyonic parameter κ determines the statistical signature of the particles.

4. Preliminary analysis

The asymptotic behaviour of the BAE (16) reveals that bound states exist for $c < 0$ and $c < 2\epsilon v/L$. In general, the form of the bound states appears to be extremely complicated. For example, for $N = 2$ and $L(c - p\epsilon) \gg 1$, we have the bound state $\{\frac{1}{2}p \pm i\frac{1}{2}(c - p\epsilon)\}$. In this communication we focus on the repulsive regime, where $c > 0$ and $|\epsilon| < c/(2p)$, for which the Bethe roots are real.

In the weak coupling limit $\gamma \ll \cos(\kappa/2)$, the leading term for the ground-state energy, obtained from the BAE (16), is

$$\frac{E_0}{N} \approx \frac{(N-1)}{L}(c' - 2\epsilon'v/L) + \frac{v^2}{L^2}. \quad (19)$$

To this order, for $\kappa = 0$ the ground-state energy is independent of the nonlinear dispersion ϵ and reduces to that of weakly interacting bosons. However, the quasimomentum distribution

$$g((k_j + k_{j+1})/2) = \frac{1}{L(k_{j+1} - k_j)} \quad (20)$$

deviates from the usual semicircle law [9, 30, 31] at $\epsilon = 0$ (see figure 1). For $\epsilon > 0$, the high density distribution drifts to the right-hand side and the front becomes steeper as the nonlinear dispersion ϵ increases. For $\epsilon < 0$, the high density part drifts to the left hand side. The nature of this kind of quantum drift in quasimomentum space is due to the delta shift or equivalently the nonlinear dispersion of velocity, which effectively increases or decreases the 1D scattering length in individual pairwise collisions.

When the total momentum is non zero, i.e. $p \neq 0$, the left and right drifts are no longer symmetric with respect to the sign of ϵ . Here the anyonic parameter κ also introduces quantum drifts in quasimomentum space. A particular example is illustrated in figure 2. For $\epsilon > 0$, the front of the distribution becomes very steep and the width becomes narrower as ϵ increases. The energy decreases as ϵ increases. However, for $\epsilon < 0$, the distribution becomes flatter. In this case, the energy increases as $|\epsilon|$ increases.

In the strong coupling limit $Lc \gg 1$, one can perform the strong coupling expansion with the BAE (16). In this limit the ground state of the model becomes that of the Tonks–Girardeau gas. This is mainly because in the strong repulsive limit, the particles behave like fermions in

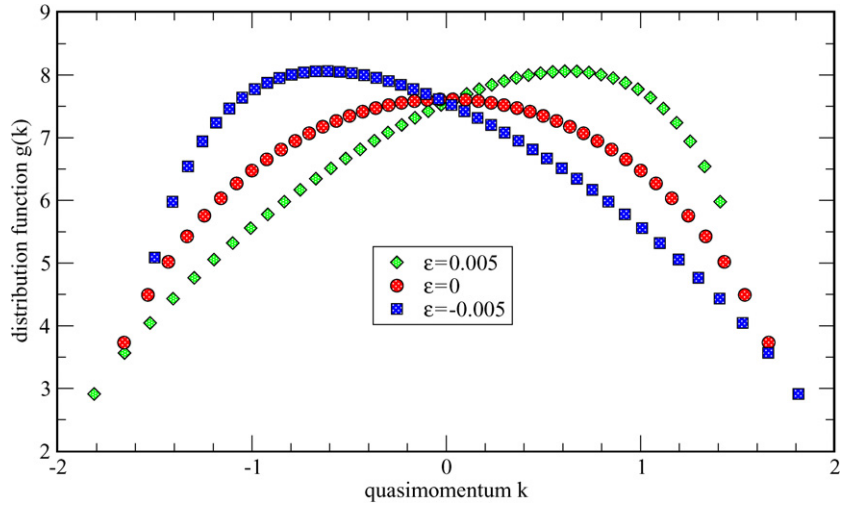


Figure 1. Typical figure showing the quantum drift effect of the nonlinear dispersion ϵ on the quasimomentum distribution function. Here $\kappa = 0$, $c = 0.04$, $L = 2$ and $N = 45$. Drifts to the left and right are symmetric with respect to the sign of ϵ .

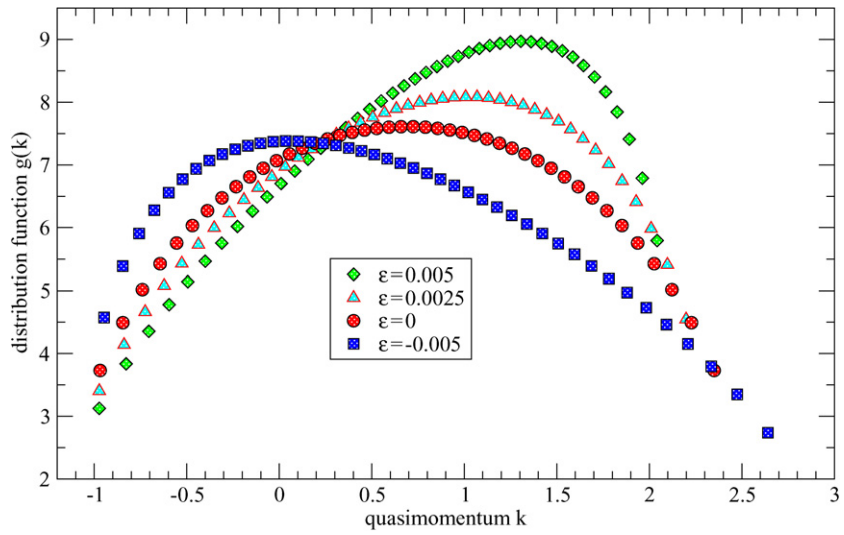


Figure 2. Typical figure showing the quantum drift effect in the quasimomentum distribution function with nonlinear dispersion ϵ and anyonic parameter κ . Here $\kappa = \pi/100$, $c = 0.04$, $L = 2$ and $N = 45$. The drifts are no longer symmetric with respect to the sign of ϵ .

thermalized states. As a result, the effect induced by the nonlinear dispersion of velocity is suppressed. To obtain the explicit result for the ground-state energy, let all quasimomenta k_j shift to $\lambda_j = k_j - v/L$ and for simplicity, choose N to be odd. In this limit, the quasimomenta are given explicitly by

$$\lambda_j \approx \left[\frac{2n_j\pi}{L} \left(1 - \frac{4n_j\epsilon'\pi}{(Lc' - \epsilon'v)^2} \right) + \frac{2\epsilon'N(N^2 - 1)\pi^2}{3L(Lc' - \epsilon'v)^2} \right] \left(1 + \frac{2N}{(Lc' - \epsilon'v)} \right)^{-1}, \quad (21)$$

where $n_j = -(N-1)/2, \dots, (N-1)/2$. This result indicates that $\sum_{j=1}^N \lambda_j = 0$. The corresponding ground-state energy is given by

$$\frac{E_0}{N} \approx \frac{(N^2-1)\pi^2}{3L^2} \left(1 + \frac{2N}{(Lc' - \epsilon'v)}\right)^{-2} + \frac{v^2}{L^2}. \quad (22)$$

It can be seen from the Bethe roots (21) that the flat fermion-like distribution becomes inclined as $|\epsilon|$ becomes large. The distribution almost linearly increases or decreases depending on the direction of the current and the parameters κ and ϵ . The system is strongly collective in the Tonks–Girardeau limit.

5. Concluding remarks

We have presented an exactly solved model of 1D anyons with general δ -function and derivative δ -function interaction. The Bethe ansatz solution has been obtained by means of the coordinate Bethe ansatz. We have seen that the anyonic signature is inextricably related to the nonlinear dispersion of velocity and the pairwise dynamical interaction between identical particles. Competing interactions among the anyonic parameter κ and the strengths of the δ -function interaction c and the nonlinear dispersion of velocity ϵ result in more subtle bound states and scattering states. Preliminary analysis of the Bethe ansatz solution in the weak and strong coupling limits has revealed drifts in the quasimomentum distribution as a function of the nonlinear dispersion and the anyonic parameter.

The quasimomentum distribution may provide a plausible way of observing anyonic behaviour in a general model with δ -function and derivative δ -function interaction. In particular, it suggests the possibility of observing anyonic behaviour via the generalized nonlinear Schrödinger equation, perhaps with regard to the propagation of solitons in nonlinear media. On the other hand, the observed quantum drifts in distribution may perhaps also be seen in experiments with trapped ultracold atoms in and out of equilibrium [32, 33]. The Bethe ansatz solution should also provide an accessible way of exploring the dynamics in the time evolution of the generalized nonlinear Schrödinger equation (3), just as the time-dependent dynamics of the 1D boson model has recently been investigated [34]. In addition, the investigation of fractional exclusion statistics [14–16, 35–38] in this generalized 1D anyon gas would provide an insight into the influence on the statistical signature of both dynamical interaction and nonlinear dispersion.

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